

On some cases of integrability of a general Riccati equation

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Abstract

A general Riccati equation is integrated in quadratures in case one of its coefficients is an arbitrary function and two others are expressed through it.

It is known that a general Riccati equation

$$y' = P(t)y^2 + Q(t)y + F(t) \quad (1)$$

can be integrated in quadratures only in special cases. These ones are given, for instance, in [1,2].

We will show that it is possible to find the solution of the equation (1) in case one of the coefficients P,Q or F is an arbitrary function and two others can be expressed through it.

Changing $y = -\frac{1}{P} \frac{u'(t)}{u(t)}$ we can write the equation (1) in the form of an equation of the second order in regards to a new variable u :

$$u'' - \left(\frac{P'}{P} + Q\right)u' + PFu = 0 \quad (2)$$

Now we consider equation (2) as a system of two equations of the first order:

$$\begin{cases} u' &= z \\ z' &= -PFu + \left(\frac{P'}{P} + Q\right)z \end{cases} \quad (3)$$

that can be rewritten as a matrix equation $\dot{X} = A(t)X$ (here $X = \begin{pmatrix} u \\ v \end{pmatrix}$) with the matrix of coefficients:

$$A(t) = \begin{pmatrix} 0 & 1 \\ -PF & \frac{P'}{P} + Q \end{pmatrix}. \quad (4)$$

It is also known [3], that if a matrix $A(t)$ is of the form

$$\begin{pmatrix} a_{11}(t) & a_{12}(t) \\ c_1 a_{12}(t) & a_{11}(t) + c_2 a_{12}(t) \end{pmatrix} \quad (5)$$

where c_1 and c_2 are arbitrary constants then $A(t)$ satisfies the functional commutativity condition

$$A(t')A(t'') - A(t'')A(t') = 0, \quad \forall t', t'' \in \mathbf{R} \quad (6)$$

and a fundamental matrix of solutions of a corresponding system is the matrix exponential $X(t) = \exp\left(\int_0^t A(\tau)d\tau\right)$ [3].

Now we prove that under a special choice of the elements of the matrix (4) it is the functionally commutative one so the system (3) is integrable and the equation (2) has an explicit solution.

Considering $A(t)$ of the form (5) we have $a_{11}(t) = 0$, $a_{12}(t) = 1$. Due to relations for elements $a_{21}(t)$ and $a_{22}(t)$ we get a system for determination of coefficients of the Riccati equation:

$$\begin{cases} c_1 &= -PF \\ c_2 &= \frac{P'}{P} + Q. \end{cases} \quad (7)$$

We will show that solving of (1) is possible when one of the coefficients F, P or Q is an arbitrary one and two others are expressed through it.

Let $F(t)$ be an arbitrary function and $c_1 \neq 0$ in (7), then we have

$$P(t) = \frac{-c_1}{F(t)}; \quad Q(t) = c_2 - \frac{P'(t)}{P(t)} = c_2 + \frac{F'(t)}{F(t)}.$$

We substitute the expressions for $P(t)$ and $Q(t)$ into (4) and find

$$A(t) = \begin{pmatrix} 0 & 1 \\ c_1 & c_2 \end{pmatrix}. \quad (8)$$

So the matrix of the system (3) turns out to be a matrix with constant coefficients and a corresponding Riccati equation (1) can be written as follows:

$$y' = -\frac{c_1}{F(t)}y^2 + (c_2 + \frac{F'(t)}{F(t)})y + F(t). \quad (9)$$

If we take $P(t)$ as an arbitrary function in (7) then functions $F(t)$ and $Q(t)$ can be immediately expressed through it and they have the form

$$F(t) = -\frac{c_1}{P(t)}; \quad Q(t) = c_2 - \frac{P'(t)}{P(t)}.$$

With these expressions for the functions we have the same form (8) of $A(t)$ and the initial Riccati equation gives over into

$$y' = P(t)y^2 + (c_2 - \frac{P'(t)}{P(t)})y - \frac{c_1}{P(t)}. \quad (10)$$

If we express $P(t)$ and $F(t)$ through $Q(t)$ then we obtain $\frac{P'(t)}{P(t)} = c_2 - Q(t)$

and $P(t) = C \exp^{c_2 t} \exp^{\int Q(t) dt}$.

For $F(t)$ we have the following equality:

$$F(t) = -\frac{c_1}{P(t)} = -\frac{c_1}{C} \exp^{-c_2 t} \exp^{\int Q(t) dt}.$$

In this case $A(t)$ also has the form (8) and the Riccati equation can be written as

$$y' = C \exp^{c_2 t} \exp^{-\int Q(t) dt} y^2 + Q(t)y - \frac{c_1}{C} \exp^{-c_2 t} \exp^{\int Q(t) dt}. \quad (11)$$

Thus, for functionally commutative matrix of the system (3) which has the form (5) there are three possible variants of the Riccati equation depending on our choice of arbitrary functions $P(t)$, $F(t)$ or $Q(t)$. All three equations are integrable in quadratures.

For instance, if $F(t)$ is an arbitrary function then the solution of (9) is of the form:

$$y(t) = \frac{((2c_1 + Cc_2) \sinh(\frac{1}{2}\sqrt{c_2^2 + 4c_1}t) + C\sqrt{c_2^2 + 4c_1} \cosh(\frac{1}{2}\sqrt{c_2^2 + 4c_1}t))F(t)}{((2C - c_2) \sinh(\frac{1}{2}\sqrt{c_2^2 + 4c_1}t) + \sqrt{c_2^2 + 4c_1} \cosh(\frac{1}{2}\sqrt{c_2^2 + 4c_1}t))c_1}$$

where C is an arbitrary constant which is determined from the initial conditions.

Letting $P(t)$ be an arbitrary function we obtain the following form of solution of (10):

$$y(t) = \frac{(2c_1 + Cc_2) \sinh(\frac{1}{2}\sqrt{c_2^2 + 4c_1}t) + C\sqrt{c_2^2 + 4c_1} \cosh(\frac{1}{2}\sqrt{c_2^2 + 4c_1}t)}{((c_2 - 2C) \sinh(\frac{1}{2}\sqrt{c_2^2 + 4c_1}t) - \sqrt{c_2^2 + 4c_1} \cosh(\frac{1}{2}\sqrt{c_2^2 + 4c_1}t))P(t)}.$$

If choose $Q(t)$ as an arbitrary function then we get the solution of (11) in the form:

$$y(t) = \exp^{-c_2 t} \exp^{\int Q(t) dt} \times \frac{((2c_1 + Cc_2) \sinh(\frac{1}{2}\sqrt{c_2^2 + 4c_1}t) + C\sqrt{c_2^2 + 4c_1} \cosh(\frac{1}{2}\sqrt{c_2^2 + 4c_1}t))}{(c_2 - 2C) \sinh(\frac{1}{2}\sqrt{c_2^2 + 4c_1}t) - \sqrt{c_2^2 + 4c_1} \cosh(\frac{1}{2}\sqrt{c_2^2 + 4c_1}t)A}.$$

where A can be assumed to equal 1 without loss of generality.

These found cases of the solution in quadratures of the general Riccati equation are different from the known ones [2] and they can be useful when solving some applied problems.

References

- [1] Matveev N. M. *Methods of Ordinary Differential Equations Integration*. M., 1963;
- [2] Kamke E. *Ordinary Differential Equations Handbook*. M., 1961;
- [3] Kovalevskaya N.M., Kovalevsky M.M. *The conditions of integrability in quadratures the second order linear systems of differential equations.*, dep. RISTI, N 2450-B00, 2000.